Rigorous Bounds on Greybody Factors: Scalar Emission of Negative Angular Momentum Modes from Myers-Perry Black Holes

Tritos Ngampitipan, Petarpa Boonserm, Auttakit Chatrabhuti, and Matt Visser

***Abstract*—When taking into account the quantum effects, a black hole can emit the so-called Hawking radiation. This Hawking radiation propagates in a curved spacetime due to the presence of a black hole. In this paper, the Myers-Perry black hole is considered, which is an uncharged, rotating black hole occurring in higher dimensions. Scalar Hawking radiation emitted from the Myers-Perry black hole is studied. The rigorous bounds on the greybody factors for massless scalar field of negative-angular-momentum modes are also derived.**

***Index Terms*—Greybody factor, hawking radiation, myers-perry black hole, rigorous bound.**

I. Introduction

The existence of black holes has been predicted by Einstein's general theory of relativity. The first solutions of the Einstein's field equation were discovered by Karl Schwarzschild. His solutions predicted the presence of Schwarzschild black holes, which are the uncharged, non-rotating black holes. The second type of black hole was obtained by solving the Einstein's field equation in conjunction with Maxwell's equation. This was done by Hans Reissner and Gunnar Nordström. Their solutions represented the Reissner-Nordström black holes, which are the charged, non-rotating black holes. The third set of solutions of the Einstein's field equation was discovered by Roy Kerr [1]. His solutions described the Kerr black holes, which are the uncharged, rotating black holes. The Kerr solutions were generalized to higher dimensions by Myers and Perry [2], [3]. Their results led to the prediction of Myers-Perry black holes, which are the uncharged, rotating black holes in higher dimensions.

When studying the quantum effects of black holes, Stephen Hawking showed that black holes can emit thermal radiation which became known as Hawking radiation [4]. The curvature of spacetime due to the presence of a black hole acts as the gravitational potential barrier. The scattering of Hawking radiation from this potential can be viewed as one-dimensional scattering problem in quantum mechanics. The term 'greybody factor' can be defined as the transmission probability.

In this paper, the rigorous bounds on the greybody factors for massless scalar field of negative-angular-momentum modes emitted from a Myers-Perry black hole will be derived.

II. Myers-Perry Spacetime

The Myers-Perry spacetime can be described by the metric [2], [3], [5]

$$ds^2=-dt^2+\frac{\Sigma}{\Delta}dr^2+\Sigma d\theta^2+(r^2+a^2)\sin^2\theta d\varphi^2 \\ +\frac{\mu}{r^{n-1}\Sigma}(dt-a\sin^2\theta d\varphi)^2+r^2\cos^2\theta d\Omega_n^2, \quad (1)$$

where

$$\Delta=r^2+a^2-\frac{\mu}{r^{n-1}} \text{ and } \Sigma=r^2+a^2\cos^2\theta. \quad (2)$$

Here $d\Omega_n^2$ is the metric on the unit n-sphere S^n which is given by

$$d\Omega_n^2=\left(\prod_{i=1}^{n-1}\sin^2\theta_i\right)d\theta_n^2. \quad (3)$$

The solutions of $\Delta(r)=0$ provide the location of the black hole event horizons. In this paper, we focus on massless scalar field emitted from the Myers-Perry black hole. The equation of motion of this scalar field can be described by the Klein-Gordon equation

$$\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\Phi\right)=0, \quad (4)$$

where

$$\sqrt{-g}=\left(\Sigma\sin\theta\right)\times\left(r^n\cos^n\theta\right)\times\left(\prod_{i=1}^{n-1}\sin^{n-i}\theta_i\right). \quad (5)$$

This Klein-Gordon equation governs how the scalar field Φ propagates in the Myers-Perry background. We use the separation of variables in this form

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$$\Phi(t,r,\theta,\varphi,\theta_1,\ldots,\theta_n)$$
$$= e^{-i\omega t+im\varphi}\,\tilde{R}_{j\ell m}(r)\,S_{\ell m}(\theta)\,Y_{jn}(\theta_1,\ldots,\theta_n), \quad (6)$$

where $S_{\ell m}(\theta)e^{im\varphi}$ are the spheroidal harmonics and $Y_{jn}(\theta_1,\ldots,\theta_n)$ are the hyper-spherical harmonics. The spheroidal harmonics satisfy

$$\left\{\frac{1}{\sin\theta\cos^n\theta}\frac{d}{d\theta}\left[\sin\theta\cos^n\theta\frac{d}{d\theta}\right]-\left(\omega a\sin\theta-\frac{m}{\sin\theta}\right)^2 -\frac{j(j+n-1)}{\cos^2\theta}+\lambda_{j\ell m}\right\}S_{\ell m}(\theta)=0 \quad (7)$$

while the hyper-spherical harmonics satisfy

$$\Delta_{S^n}Y_{jn}(\theta_1,\ldots,\theta_n)+j(j+n-1)Y_{jn}(\theta_1,\ldots,\theta_n)=0, \quad (8)$$

where Δ_{S^n} is the Laplacian. Then, the radial Teukolsky equation is obtained [6]-[8]

$$\left[\frac{d^2}{dr_*^2}-U_{j\ell m}(r)\right]R_{j\ell m}(r)=0, \quad (9)$$

where the tortoise coordinate r_* is defined by

$$dr_*=\frac{r^2+a^2}{\Delta(r)}dr. \quad (10)$$

The relationship between the tortoise coordinate and the ordinary coordinate is plotted as shown in Fig. 1.

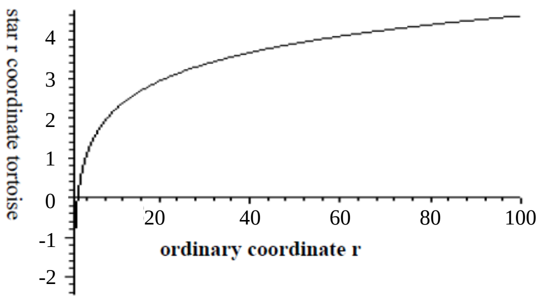

Fig. 1. Tortoise coordinate as a function of ordinary coordinate.

Here the Teukolsky potential $U_{j\ell m}(r)$ is given by [5]

$$U_{j\ell m}(r)=\frac{\Delta(r)}{\left(r^2+a^2\right)^2}\left[\lambda_{j\ell m}+\frac{j(j+n-1)a^2}{r^2}+\frac{n(n-2)\Delta(r)}{4r^2}+\frac{n\Delta'(r)}{2r}-\frac{3r^2\Delta(r)}{(r^2+a^2)^2}+\frac{[r\Delta(r)]'}{r^2+a^2}\right]-\left(\omega-m\varpi\right)^2, \quad (11)$$

where

$$\varpi=\frac{a}{r^2+a^2}. \quad (12)$$

This Teukolsky potential can be expressed in another form as

$$U_{j\ell m}(r)=V_{j\ell m}(r)-(\omega-m\varpi)^2, \quad (13)$$

where

$$V_{j\ell m}(r)=\frac{\Delta(r)}{\left(r^2+a^2\right)^2}\left[\lambda_{j\ell m}+\frac{j(j+n-1)a^2}{r^2}+\frac{n(n-2)\Delta(r)}{4r^2}+\frac{n\Delta'(r)}{2r}-\frac{3r^2\Delta(r)}{(r^2+a^2)^2}+\frac{[r\Delta(r)]'}{r^2+a^2}\right]. \quad (14)$$

The potential $V_{j\ell m}(r)$ is plotted as shown in Fig. 2 for five and six dimensions which correspond to n = 1 and n = 2, respectively.

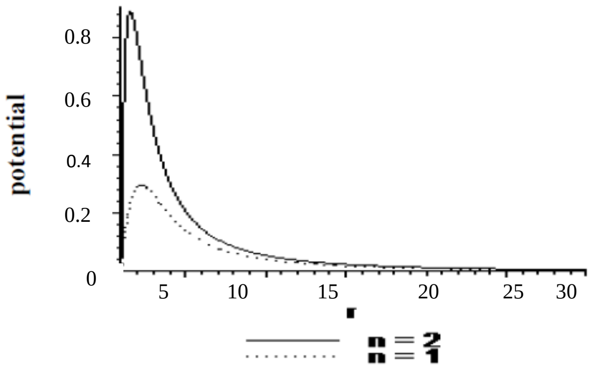

Fig. 2. The potential $V_{j\ell m}(r)$ for n = 1 and n = 2.

III. Rigorous Bounds on Greybody Factors

We can model the scattering of the massless scalar field from the Teukolsky potential as one-dimensional scattering problem in quantum mechanics. The term 'greybody factor' in black hole systems can be defined as the 'transmission probability'. In general situations, finding exact greybody factors is difficult due to complicated potentials. Therefore, in this paper, some rigorous bounds will be placed on greybody factors. These bounds were first developed in [9]. Their further developments can be found in [10]-[13] and their applications can be found in [14]-[19]. For the radial Teukolsky equation in (9), the rigorous bounds on the greybody factors are given by

$$T_{j\ell m}\geq \operatorname{sech}^2\left[\int_{-\infty}^{\infty}\frac{\left|\tilde{h}'(r_*)\right|}{2\tilde{h}(r_*)}dr_*+\int_{-\infty}^{\infty}\frac{\left|\tilde{U}_{j\ell m}(r_*)+\tilde{h}^2(r_*)\right|}{2\tilde{h}(r_*)}dr_*\right], \quad (15)$$

for any positive functions $h(r_*)$. In this paper, we choose

$$\tilde{h}(r_*)=h(r)=\omega-m\varpi(r), \quad (16)$$

where $m < 0$. In this case, we obtain

$$\tilde{h}'(r_*) = \frac{2mar\Delta(r)}{\left(r^2+a^2\right)^3} > 0\,. \tag{17}$$

Then, we obtain the first integral

$$\int_{-\infty}^{\infty} \frac{|\tilde{h}'(r_*)|}{2\tilde{h}(r_*)} dr_* = \frac{1}{2}\ln\frac{\tilde{h}(\infty)}{\tilde{h}(-\infty)} = \frac{1}{2}\ln(1-m/m_*)\,, \tag{18}$$

where

$$m_* = \frac{\omega}{\Omega_H}\,,\ \Omega_H = \frac{a}{a^2+r_H^2}\,, \tag{19}$$

and r_H is the event horizon radius Since $\omega - m\Omega_H > h(r) > \omega$, we have an inequality

$$\int_{-\infty}^{\infty} \frac{|U_{j\ell m}+h^2(r)|}{2h(r)} dr_* = \int_{-\infty}^{\infty} \frac{|V_{j\ell m}|}{2h(r)} dr_* < \int_{-\infty}^{\infty} \frac{V_{j\ell m}}{2\omega} dr_*\,. \tag{20}$$

Using (14), we can write

$$\int_{-\infty}^{\infty} \frac{V_{j\ell m<0}}{2\omega} dr_* = \frac{1}{2\omega}\int_{-\infty}^{\infty} \frac{\Delta(r)}{\left(r^2+a^2\right)^2}\left[\lambda_{j\ell m<0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{[r\Delta(r)]'}{r^2+a^2}\right] dr_*\,. \tag{21}$$

Using (10), we can change the variable r_* to r

$$\int_{-\infty}^{\infty} \frac{V_{j\ell m<0}}{2\omega} dr_* = \frac{1}{2\omega}\int_{r_H}^{\infty} \frac{\Delta(r)}{\left(r^2+a^2\right)^2}\left[\lambda_{j\ell m<0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{[r\Delta(r)]'}{r^2+a^2}\right] \frac{r^2+a^2}{\Delta(r)} dr\,. \tag{22}$$

The above equation can be simplified to

$$\int_{-\infty}^{\infty} \frac{V_{j\ell m<0}}{2\omega} dr_* = \frac{1}{2\omega}\int_{r_H}^{\infty} \frac{1}{r^2+a^2}\left[\lambda_{j\ell m<0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{[r\Delta(r)]'}{r^2+a^2}\right] dr\,. \tag{23}$$

Therefore,

$$T_{j\ell,m<0} \ge \operatorname{sech}^2\left[\frac{1}{2}\ln(1-m/m_*) + \frac{1}{2\omega r_H} I_{j\ell,m<0}\right], \tag{24}$$

where

$$I_{j\ell m<0} = \frac{n(2n-3)}{8} + j(j+n-1) + \frac{a^2}{4(r_H^2+a^2)} + \left(\frac{2n+1}{2} - j(j+n-1) + \lambda_{j\ell m}(a\omega)\right)\frac{r_H}{a}\arctan\frac{a}{r_H} + \frac{n(r_H^2+a^2)}{8r_H^2}\,{}_2F_1\left(1,\frac{n+2}{2},\frac{n+4}{2},-\frac{a^2}{r_H^2}\right). \tag{25}$$

Here the hypergeometric function ${}_2F_1(a, b, c, z)$ is defined by

$$ {}_2F_1(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}\,. \tag{26}$$

The bounds on the greybody factors are plotted as shown in Fig. 3.

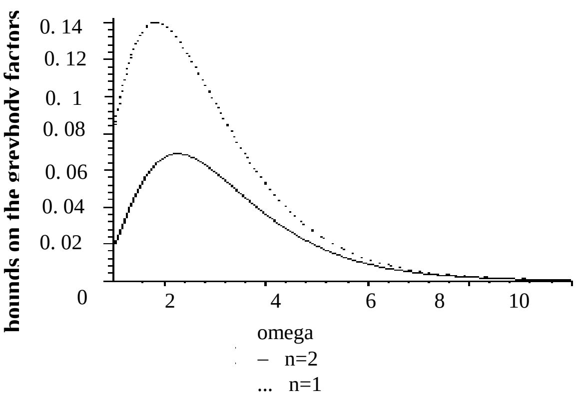

Fig. 3. The bounds on the greybody factors for $n = 1$ and $n = 2$.

In the limit $a \to 0$ and $n = j = 0$, the quantity $I_{j\ell m<0}$ reduces to

$$I_{j=0,\ell,m<0} = \frac{1}{2} + \lambda_{j=0,\ell,m<0}\,, \tag{27}$$

which is the result for the Schwarzschild black hole [14].

IV. Conclusion

In this paper, the rigorous bounds on the greybody factors for massless scalar field of negative-angular-momentum modes emitted from the Myers-Perry black hole have been established. To obtain these bounds, the appropriate function $h(r_*)$ has been chosen. The number of dimensions of spacetime, the angular momentum of the black holes, and the mass of the black hole have been determined to have effects on these bounds. Note that for $n = 0$, these bounds reduce to bounds for Kerr black holes. For outlook, we can choose other forms of $h(r_*)$ in order to derive better bounds.

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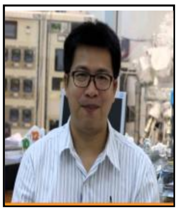

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