# Conservation of Flux in Superradiance Phenomenon

Petarpa Boonserm, Tritos Ngampitipan, and Matt Visser

***Abstract*—For a usual occurrence of wave scattering, the amplitude of the reflected wave is less than that of the incident wave because the incident wave loses energy to the reflective obstacle. However, for the so-called superradiance phenomenon, the amplitude of the reflected wave is more than that of the incident wave since the incident wave extracts energy from the reflective obstacle. In this paper, a simple toy model of superradiance is presented. The results show that for the case of superradiance, we derive a conservation of flux instead of the conservation of probability.**

***Index Terms*—Conservation, flux, probability, superradiance.**

## I. Introduction

The phenomena of scattering can be described by the interaction of wave with a reflective physical obstacle. In a general situation, the incident wave loses some of its energy to the obstacle, resulting in the amplitude of the reflected wave being less than that of the incident wave. However, in some systems, the incident wave gains energy from the obstacle instead of losing energy. Therefore, the amplitude of the reflected wave becomes greater than that of the incident wave. This unusual phenomenon is called superradiance. Matters of superradiance in literature can be found in [1]-[20].

Despite a long scientific history, superradiance still generates some degree of confusion. Part of the confusion comes from a lack of understanding of the differences between fluxes and probabilities. In this paper, a simple toy model of superradiance is presented to clarify the concept.

## II. Superradiance

In non-relativistic quantum mechanics, superradiance does not take place [21]. To see this, consider the Schrödinger equation

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\psi(x,t) + V(x)\psi(x,t). \tag{1}$$

Assuming the solution

$$\psi(x,t) = e^{-i\omega t/\hbar}\psi(x), \tag{2}$$

The Schrödinger equation becomes

$$\frac{\hbar^2}{2m}\partial_x^2\psi(x) = [V(x)-\omega]\psi(x). \tag{3}$$

On the other hand, in the relativistic regime we have the Klein-Gordon equation

$$\left\lfloor -(\partial_t - i\varpi(x))^2 + \partial_x^2 - V(x) \right\rfloor \psi(t,x) = 0. \tag{4}$$

For a neutral scalar field, we assume the solution

$$\psi(x,t) = e^{-i\omega t}\psi(x). \tag{5}$$

The Klein-Gordon equation becomes

$$\partial_x^2\psi(x) = [V(x) - \{\omega - \varpi(x)\}^2]\psi(x). \tag{6}$$

In this case, superradiance can occur. We see that the term $[\omega-\varpi(x)]^2$ is responsible for superradiance. For a charged scalar field, we obtain

$$\partial_x^2\psi(x) = [V(x) - \{\omega - \varpi(x) - q\Phi(x)\}^2]\psi(x), \tag{7}$$

where $q$ is the charge of the scalar field. The term $[\omega-\varpi(x)-q\Phi(x)]^2$ is also responsible for superradiance.

## III. Fluxes in Superradiance Phenomenon

In ordinary phenomena of wave scattering, we are familiar with the term ‘probability’ through both ‘reflection probability’ and ‘transmission probability’. For a more general situation, including the case of superradiance, it is preferable to calculate the quantities in terms of fluxes rather than probabilities. The general conservation law can be described by

$$F_{\text{reflected}} + F_{\text{transmitted}} = 1 - F_{\text{dissipated}}. \tag{8}$$

In this paper, we are interested in cases of non-dissipation, where $F_{\text{dissipated}} = 0$. The general cases, including dissipation, can be found in [21]. In ordinary cases, if the transmitted flux is non-negative $F_{\text{transmitted}} \geq 0$, it can be reduced to transmission probability $F_{\text{transmitted}} = T$. Moreover, the reflected flux also reduces to reflection probability $F_{\text{reflected}} = R$. Therefore (8) becomes

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$$R+T=1. \tag{9}$$

This is the familiar conservation law of probabilities. On the other hand, in the case of superradiance, we have $F_{\text{transmitted}}<0$. It cannot be interpreted as the transmission probability. Thus, in any situation, we should work with quantities in terms of fluxes rather than probabilities.

## IV. Toy Model for Superradiance

Consider the Klein-Gordon equation in 1+1 dimensions

$$\left\lfloor -(\partial_t - i\varpi(x))^2 + c^2\partial_x^2 - V(x)\right\rfloor \psi(t,x)=0. \tag{10}$$

Assuming the solution $\psi(t,x)=e^{-i\omega t}\psi(x)$, we obtain

$$c^2\,\partial_x^2\psi(x)=[V(x)-\{\omega-\varpi(x)\}^2]\psi(x). \tag{11}$$

Now, we simplify the problem by letting $V(x)\to 0$ and taking

$$\varpi(x)=\Omega\mathrm{sign}(x), \tag{12}$$

where $\Omega$ is a constant. Moreover, we set $c=1$. Therefore, (11) becomes

$$\partial_x^2\psi(x)=-[\omega-\Omega\mathrm{sign}(x)]^2\psi(x). \tag{13}$$

The solutions to (13) are given by

$$\psi(x)=\begin{cases} e^{ik_-x}+re^{-ik_-x} & \text{for} \quad x<0 \\ te^{ik_+x} & \text{for} \quad x>0 \end{cases}, \tag{14}$$

where $r$ is the reflection amplitude, $t$ is the transmission amplitude, and

$$k_\pm^2=(\omega\mp\Omega)^2. \tag{15}$$

Note that

$$k_+k_-=\omega^2-\Omega^2. \tag{16}$$

Thus, we obtain

$$\mathrm{sign}(k_+k_-)=\mathrm{sign}(\omega^2-\Omega^2). \tag{17}$$

Assuming that wave moves from left to right and crosses the border at the origin, we have

$$e^{ik_-x}+re^{-ik_-x}\check{s}\quad te^{ik_+x}. \tag{18}$$

The continuity of the wave function leads to

$$1+r=t. \tag{19}$$

The continuity of the derivative of the wave function leads to

$$k_-(1-r)=k_+t. \tag{20}$$

Solving the equations, we obtain

$$1+r=\frac{k_-}{k_+}(1-r). \tag{21}$$

Rearranging it gives

$$r=-\frac{k_+-k_-}{k_++k_-}=-\frac{(\omega-\Omega)-(\omega+\Omega)}{(\omega-\Omega)+(\omega+\Omega)}=+\frac{\Omega}{\omega}. \tag{22}$$

Since the reflection amplitude is normalization independent, the result is valid. The reflected flux is given by

$$F_{\text{reflected}}=|r|^2=\frac{\Omega^2}{\omega^2}. \tag{23}$$

However, the transmission amplitude depends on the normalization. For the relativistic Klein-Gordon equation, the normalization factor is

$$\frac{e^{ik_-x}}{\sqrt{2|k_-|}}. \tag{24}$$

Therefore, the normalized solutions to (13) are given by

$$\psi(x)=\begin{cases} \dfrac{e^{ik_-x}}{\sqrt{2|k_-|}}+\dfrac{re^{-ik_-x}}{\sqrt{2|k_-|}} & \text{for} \quad x<0 \\ \dfrac{te^{ik_+x}}{\sqrt{2|k_+|}} & \text{for} \quad x>0 \end{cases}. \tag{25}$$

The continuity of the wave function leads to

$$\frac{1+r}{\sqrt{2|k_-|}}=\frac{t}{\sqrt{2|k_+|}}. \tag{26}$$

The continuity of the derivative of the wave function leads to

$$\frac{k_-}{\sqrt{2|k_-|}}(1-r)=\frac{k_+}{\sqrt{2|k_+|}}t. \tag{27}$$

Solving the equations, we obtain

$$\frac{1+r}{\sqrt{2|k_-|}}=\frac{k_-}{k_+}\frac{1-r}{\sqrt{2|k_-|}}. \tag{28}$$

Rearranging it gives

$$r=-\frac{k_+-k_-}{k_++k_-}=-\frac{(\omega-\Omega)-(\omega+\Omega)}{(\omega-\Omega)+(\omega+\Omega)}=+\frac{\Omega}{\omega}. \tag{29}$$

Substituting in (26), we obtain

$$t=\sqrt{\frac{|k_+|}{|k_-|}}\left(1+\frac{\Omega}{\omega}\right)=\sqrt{\frac{|\omega-\Omega|}{|\omega+\Omega|}}\left(\frac{\omega+\Omega}{\omega}\right). \quad (27)$$

The reflected flux is given by

$$F_{\text{reflected}}=|r|^2=\frac{\Omega^2}{\omega^2}. \quad (28)$$

If $|\omega|>|\Omega|$, we have

$$t=\sqrt{\frac{\omega-\Omega}{\omega+\Omega}}\left[\frac{\omega+\Omega}{\omega}\right]=\frac{\sqrt{\omega^2-\Omega^2}}{\omega}$$

$$=\text{sign}(\omega)\sqrt{1-\frac{\Omega^2}{\omega^2}}. \quad (29)$$

Therefore, the transmitted flux is given by

$$|t|^2=1-\frac{\Omega^2}{\omega^2}\geq 0. \quad (30)$$

We see that

$$F_{\text{reflected}}+|t|^2=1. \quad (31)$$

In this case, we can write

$$F_{\text{transmitted}}=|t|^2\geq 0. \quad (32)$$

On the other hand, if $|\omega|<|\Omega|$, we have

$$t=\sqrt{-\frac{(\omega-\Omega)}{(\omega+\Omega)}}\left(\frac{\omega+\Omega}{\omega}\right)=\frac{\sqrt{\Omega^2-\omega^2}}{\omega}$$

$$=\text{sign}(\omega)\sqrt{\frac{\Omega^2}{\omega^2}-1}. \quad (33)$$

The transmitted flux is given by

$$|t|^2=\frac{\Omega^2}{\omega^2}-1. \quad (34)$$

We see that

$$F_{\text{reflected}}-|t|^2=1. \quad (35)$$

In this case, we can write

$$F_{\text{transmitted}}=-|t|^2\leq 0. \quad (36)$$

We summarize both the cases by

$$F_{\text{transmitted}}=\text{sign}(k_+k_-)|t|^2=1-\frac{\Omega^2}{\omega^2}. \quad (37)$$

Thus, we can write

$$F_{\text{reflected}}+F_{\text{transmitted}}=1. \quad (38)$$

Using (17), this can be rewritten as

$$|r|^2+\text{sign}(k_+k_-)|t|^2=1. \quad (39)$$

Explicitly, this is not a conservation of probability, but rather, a conservation of flux.

## V. Conclusion

Superradiance is a phenomenon of scattering in which the amplitude of the reflected wave is more than that of the incident wave because the incident wave extracts energy from the reflective obstacle. In this paper, a simple toy model of superradiance has been presented. In the case of superradiance, we have achieved the conservation of flux instead of the conservation of probability. The concept of conservation of probability is only valid in the absence of superradiance. So, in any situation (both with and without superradiance) we can write the conservation of flux

$$F_{\text{reflected}}+F_{\text{transmitted}}=1 \quad (40)$$

if there is no dissipation. This can be rewritten as

$$|r|^2+\text{sign}(k_+k_-)|t|^2=1. \quad (41)$$

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