

Standard Deviation of the Run Length (SDRL) and Average Run Length (ARL) Performances of EWMA and Synthetic Charts

Michael B. C. Khoo, S. Y. Teh, X. Y. Chew, and W. L. Teoh

Abstract—Statistical process control (SPC) techniques are applied to monitor a process. The control chart is a valuable tool in SPC. The shewhart control chart was the first chart proposed in the literature of SPC and it is still used in process monitoring in today's manufacturing and service industries. Both the exponentially weighted moving average (EWMA) and synthetic charts outperform the shewhart control chart for detecting shifts in the process mean. Since both the EWMA and synthetic charts provide better process mean shifts detection performance, a study on which chart to use in process monitoring under different situations is the aim of this work. This paper compares the average run length (ARL) and standard deviation of the run length (SDRL) profiles of the EWMA and synthetic charts. Comparisons are made based on the normality assumption. The mathematica programs are used to compute the ARLs and SDRLs of the EWMA and synthetic charts. The ARL results indicate that the EWMA chart is superior to the synthetic chart for detecting small mean shifts, but the latter prevails for detecting moderate and large shifts. However, in terms of the SDRL, the EWMA chart surpasses the synthetic chart for small and moderate shifts.

Index Terms—Average run length (ARL), exponentially weighted moving average (EWMA) chart, synthetic chart, standard deviation of the run length (SDRL).

I. INTRODUCTION

The control chart which was originally developed at the Bell Laboratories by Dr Walter Shewhart [1] in 1924 is one of the primary tools in SPC. Since then, numerous research works were made on various types of control charts which include univariate and multivariate charts [2]. Two of the univariate charts considered in this paper are the EWMA and synthetic charts. The performance of a control chart is typically measured in terms of the ARL and SDRL. The ARL is the average number of sample points that is plotted on a chart before the first out-of-control signal is detected whereas the SDRL measures the spread of the run length distribution. When a process is out-of-control, it is desirable to have small values of ARL and SDRL. Sections II and III review the EWMA and synthetic charts, respectively. Sections IV and V studies the ARL and SDRL performances

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of the EWMA and synthetic charts, respectively, whereas Section VI compares the ARL and SDRL performances of these two charts. Finally, conclusions are drawn in Section VII.

II. EWMA CHART

The EWMA chart was introduced by Roberts [3] and further studied by Crowder [4]-[6], and Lucas and Saccucci [7]. The EWMA chart is a good alternative to the Shewhart \bar{X} chart for detecting small shifts [2], [3]. The EWMA statistic is defined as follows:

$$Z_i = \lambda \bar{X}_i + (1 - \lambda) Z_{i-1} \quad (1)$$

where $0 < \lambda \leq 1$ is the smoothing constant. The starting value is the process target, i.e. $Z_0 = \mu_0$. The control limits (UCL/LCL) and center line (CL) for the EWMA chart are defined as follows [2], [3]:

$$UCL = \mu_0 + L\sigma \sqrt{\frac{\lambda}{n(2-\lambda)} [1 - (1-\lambda)^{2i}]}, \quad (2a)$$

$$CL = \mu_0, \text{ and} \quad (2b)$$

$$LCL = \mu_0 - L\sigma \sqrt{\frac{\lambda}{n(2-\lambda)} [1 - (1-\lambda)^{2i}]}, \quad (2c)$$

where the factor L controls the width of the control limits. An out-of-control is signaled by the EWMA chart when Z_i plots beyond the limits UCL/LCL . In this paper, the ARL of the EWMA chart is computed using the Markov chain approach presented in Zhang *et al.* [8]. The Markov chain procedure involves dividing the interval between the UCL and LCL into $t = 2m + 1$ subintervals, each of width $2d$. The EWMA statistic, Z_i is said to be in a transient state j at time i if $S_j - d < Z_i < S_j + d$, for $j = -m, -m + 1, \dots, 0, \dots, m - 1, m$, where S_j represents the midpoint of the j^{th} subinterval. The run length of the EWMA chart is represented by its initial probability vector and transition probability matrix. The initial probability vector is

$$\mathbf{u} = (0, \dots, 0, 1, 0, \dots, 0)', \quad (3)$$

Here, the entry having the value unity corresponds to the initial state of the Markov chain, where the $(m + 1)^{\text{th}}$ entry of \mathbf{u} has the value unity.

The ARL of the EWMA chart is computed as [8]

$$ARL = \mathbf{u}'(\mathbf{I} - \mathbf{R})^{-1} \mathbf{1}, \quad (4)$$

while its SDRL is obtained using the following formula:

$$SDRL = \sqrt{2\mathbf{u}'(\mathbf{I} - \mathbf{R})^{-2} \mathbf{R} \cdot \mathbf{1} - (ARL)^2} + ARL, \quad (5)$$

where \mathbf{u} denotes a vector of ones, \mathbf{I} represents the identity matrix whereas \mathbf{R} is the transition probability matrix for the transient states. The details for computing \mathbf{R} is given in Zhang *et al.* [8].

III. SYNTHETIC CHART

The synthetic chart which comprises the \bar{X}/S and CRL/S sub-charts was suggested by Wu and Spedding [9]. The implementation of the synthetic chart is as follows:

Step 1: Compute the \bar{X}/S sub-chart's upper and lower control limits as follows:

$$UCL_{\bar{x}/s} = \mu + k\sigma_{\bar{x}}, \quad (6a)$$

and

$$LCL_{\bar{x}/s} = \mu - k\sigma_{\bar{x}}, \quad (6b)$$

where μ and $\sigma_{\bar{x}}$ are the in-control mean and standard deviation of the sample mean \bar{X} , respectively. The factor k controls the width of the control limits.

Step 2: Compute the CRL/S sub-chart's lower control limit, L' .

Step 3: At each inspection point, take a random sample of n observations, X_i , for $i=1, 2, \dots, n$, and calculate the sample mean, \bar{X} .

Step 4: If $LCL_{\bar{x}/s} < \bar{X} < UCL_{\bar{x}/s}$, the sample is conforming and the control flow returns to Step 3. Otherwise, the sample is non-conforming and the control flow goes to the next step.

Step 5: Count the number of samples between the current and the last non-conforming samples. This number is taken as the CRL value of the CRL/S sub-chart.

Step 6: If $CRL > L'$, the process is in-control and the control flow goes back to Step 3. Otherwise, the synthetic chart signals an out-of-control condition. Corrective actions are taken to identify and remove the assignable cause(s). Then the control flow returns to Step 3.

The ARL of the synthetic chart is computed as follows:

$$ARL = \frac{1}{P} \times \frac{1}{1 - (1 - P)^{L'}}, \quad (7a)$$

where

$$P = 1 - \left[(k - \delta\sqrt{n}) - (-k - \delta\sqrt{n}) \right]. \quad (7b)$$

Here, δ is the size of the standardized mean shift, i.e. $\delta = \frac{|\mu_1 - \mu_0|}{\sigma}$, where μ_1 is the out-of-control mean and μ_0 is the in-control mean. When $\delta = 0$, the process is in-control.

IV. ARL AND SDRL PERFORMANCES OF THE EWMA CHART

The optimal parameters λ and L are obtained by minimizing the out-of-control ARL for a desired shift of interest $\delta_{opt} = 1$ so that an in-control ARL (ARL_0) of 370 is attained, based on the sample sizes $n = 3, 5, 7$, and 10 . The optimal design procedure is based on the Markov chain approach described in Zhang *et al.* [8]. From these optimal λ and L values obtained, the ARL profiles for the entire magnitude of shifts can be computed.

TABLE I: ARLS AND SDRLS OF THE EWMA CHART FOR $n = 3, 5, 7$, AND 10 , WHEN $\delta_{opt} = 1$, BASED ON $ARL_0 = 370$

n	3	5	7	10	3	5	7	10
λ	0.31	0.44	0.59	0.75	0.31	0.44	0.59	0.75
L	0.735	0.715	0.731	0.734	0.735	0.715	0.731	0.734
δ	ARL				SDRL			
0.00	371.9	370.2	372.5	372.9	369.1	368.4	371.2	369.7
0.30	63.4	50.8	48.6	16.8	59.8	48.2	46.8	13.1
0.50	14.9	10.6	9.1	4.8	11.6	8.3	7.4	2.3
0.80	6.7	4.6	3.6	2.8	4.0	2.7	2.3	1.0
1.00	4.2	2.8	2.2	2.0	2.0	1.3	1.1	0.6
1.50	2.4	1.7	1.3	1.3	0.8	0.6	0.5	0.5
2.00	1.8	1.2	1.0	1.0	0.6	0.4	0.1	0.1
3.00	1.1	1.0	1.0	1.0	0.3	0.0	0.0	0.0
4.00	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0

Table I shows the ARL and $SDRL$ profiles for $0 \leq \delta \leq 4$. It is obvious that both ARL and $SDRL$ decrease as δ increases. This indicates that larger shifts can be detected quicker and will result in smaller spread in the run length distribution. The results also show that as the sample size increases from $n = 3$ to $n = 10$, the ARL s and $SDRL$ s of the EWMA chart reduce, for the same size of a mean shift, δ .

V. ARL AND SDRL PERFORMANCES OF THE SYNTHETIC CHART

TABLE II: ARLS AND SDRLS OF THE SYNTHETIC CHART FOR $n = 3, 5, 7$, AND 10 , WHEN $\delta_{opt} = 1$, BASED ON $ARL_0 = 370$

n	3	5	7	10	3	5	7	10
L'	6	4	3	2	6	4	3	2
k	2.294	2.219	2.164	2.085	2.294	2.219	2.164	2.085
δ	ARL				SDRL			
0.00	370.0	370.0	370.0	370.0	409.8	403.2	399.2	394.2
0.50	33.6	16.6	10.2	6.3	41.8	20.7	12.6	7.5
0.75	9.6	4.5	2.9	1.9	12.1	5.3	3.0	1.8
1.00	4.0	2.1	1.5	1.2	4.4	1.8	1.0	0.6
1.50	1.6	1.1	1.0	1.0	1.0	0.4	0.2	0.1
2.00	1.1	1.0	1.0	1.0	0.4	0.1	0.0	0.0
3.00	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0
4.00	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0

To ensure a fair comparison, the optimal L' and k values of the synthetic chart are determined so that $ARL_0 = 370$, $\delta_{opt} = 1$ and $n = 3, 5, 7$, and 10 . The optimization procedure given in Wu and Spedding [9] is used to compute the optimal parameters L' and k . Then using the optimal parameters L' and k , the ARL of the synthetic chart is computed using (7a) and (7b), while its $SDRL$ is obtained

using (5) but by substituting u , R , and ARL with that corresponding to the synthetic chart [10].

For example, for $ARL_0 = 370$, $\delta_{opt} = 1$, and $n = 3$, the optimal parameters of the synthetic chart are computed as $L' = 6$ and $k = 2.294$. Table II indicates that as δ increases (size of a mean shift increases), the ARLs and SDRLs decrease. The findings also show that as the sample size, n , increases, the ARLs and SDRLs decrease for the same magnitude of a mean shift, δ .

VI. A COMPARISON OF THE ARL AND SDRL PERFORMANCES OF THE EWMA AND SYNTHETIC CHARTS

TABLE III: ARLS OF THE EWMA AND SYNTHETIC CHARTS, WHERE $n = 3, 5, 7$, AND 10 , $\delta_{opt} = 1$, AND $ARL_0 = 370$

n	3	5	7	10	3	5	7	10
δ	EWMA				Synthetic			
0.00	371.9	370.2	372.5	372.9	370.0	370.0	370.0	370.0
0.25	63.4	50.8	48.6	16.8	151.8	102.8	75.4	52.8
0.50	14.9	10.6	9.1	4.8	33.6	16.6	10.2	6.3
0.75	6.7	4.6	3.6	2.8	9.6	4.5	2.9	1.9
1.00	4.2	2.8	2.2	2.0	4.0	2.1	1.5	1.2
1.50	2.4	1.7	1.3	1.3	1.6	1.1	1.0	1.0
2.00	1.8	1.2	1.0	1.0	1.1	1.0	1.0	1.0
3.00	1.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0
4.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

TABLE IV: SDRLS OF THE EWMA AND SYNTHETIC CHARTS, WHERE $n = 3, 5, 7$, AND 10 , $\delta_{opt} = 1$, AND $ARL_0 = 370$

n	3	5	7	10	3	5	7	10
δ	EWMA				Synthetic			
0.00	369.1	368.4	371.2	369.7	409.8	403.2	399.2	394.2
0.25	59.8	48.2	46.8	13.1	175.0	118.3	86.6	60.3
0.50	11.6	8.3	7.4	2.3	41.8	20.7	12.6	7.5
0.75	4.0	2.7	2.3	1.0	12.1	5.3	3.0	1.8
1.00	2.0	1.3	1.1	0.6	4.4	1.8	1.0	0.6
1.50	0.8	0.6	0.5	0.5	1.0	0.4	0.2	0.1
2.00	0.6	0.4	0.1	0.1	0.4	0.1	0.0	0.0
3.00	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Combining the ARLs and SDRLs in Table I and Table II give the results in Table III and Table IV. Table III clearly shows that the ARL produced by the EWMA chart is smaller than that of the synthetic chart when $0 < \delta \leq 0.75$. However, the synthetic chart gives smaller ARL than the EWMA chart when $\delta \geq 1$. Table IV shows that the EWMA chart outperforms the synthetic chart for $\delta \leq 1.5$, in terms of the SDRL. The ARL results show that the EWMA chart is superior to the synthetic chart for detecting small mean shifts, but the latter prevails for detecting moderate and large shifts. However, in terms of the SDRL, the EWMA chart surpasses the synthetic chart for small and moderate δ values. This study guides industrial practitioners to choose between the EWMA and synthetic charts. For example, if past experience indicates that a small mean shift is likely to occur, then the EWMA chart should be used to monitor the process. However, if historical data show that a moderate or large shift usually happens, then the synthetic chart should be employed in process monitoring. Overall, similar trends are observed for $n = 3, 5, 7$, and 10 , as displayed in Table III to Table IV, respectively. It should be

noted that considering other combinations of δ_{opt} and ARL_0 will give similar outcomes. Therefore, it suffices to only discuss the results for $\delta_{opt} = 1$, $ARL_0 = 370$, and $n = 3, 5, 7$ and 10 . The results for $\delta_{opt} = 1$, $ARL_0 = 250$, and $n = 3, 5, 7$ and 10 which correspond to Tables I to IV are given in the Appendix (see Tables A.I to A.IV).

VII. CONCLUSION

In this paper, the ARLs and SDRLs of the EWMA and synthetic charts are compared for a process having a normal distribution. This preliminary study provides some insight on EWMA and synthetic charts to practitioners to facilitate process monitoring, where a proper and careful selection of the correct chart to use is of utmost importance so that a delay in detecting process shifts will not happen. Further research should compare the EWMA and synthetic charts using other performance criteria, such as the median run length (MRL) and percentiles of the run length distribution. It is also interesting to compare these charts when process parameters, i.e. the target mean and standard deviation are estimated from a Phase-I dataset. In most real situations, process parameters have to be estimated as they are rarely known. Another topic for future work involves comparing the multivariate synthetic and multivariate EWMA (MEWMA) charts when the process parameters are both known and unknown. Research can also be made when the underlying process has a skewed distribution or when the independence assumption of the data is violated.

APPENDIX

TABLE A.I: ARLS AND SDRLS OF THE EWMA CHART FOR $n = 3, 5, 7$, AND 10 , WHEN $\delta_{opt} = 1$, BASED ON $ARL_0 = 250$

n	3	5	7	10	3	5	7	10
λ	0.34	0.48	0.63	0.78	0.34	0.48	0.63	0.78
L	0.735	0.716	0.735	0.727	0.735	0.716	0.735	0.727
δ	ARL				SDRL			
0.00	252.0	250.5	250.5	250.2	249.6	248.9	249.5	249.4
0.30	52.6	42.7	40.1	36.3	49.5	40.5	38.6	35.2
0.50	13.5	9.7	8.3	6.7	10.6	7.7	6.8	5.7
0.80	6.2	4.3	3.4	2.6	3.8	2.6	2.2	1.7
1.00	3.9	2.6	2.0	1.5	1.9	1.3	1.0	0.8
1.50	2.3	1.5	1.2	1.0	0.8	0.6	0.4	0.2
2.00	1.7	1.1	1.0	1.0	0.6	0.3	0.1	0.0
3.00	1.1	1.0	1.0	1.0	0.3	0.0	0.0	0.0
4.00	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0

TABLE A.II: ARLS AND SDRLS OF THE SYNTHETIC CHART FOR $n = 3, 5, 7$, AND 10 , WHEN $\delta_{opt} = 1$, BASED ON $ARL_0 = 250$

n	3	5	7	10	3	5	7	10
L'	6	4	3	2	6	4	3	2
k	2.216	2.139	2.084	2.002	2.216	2.139	2.084	2.002
δ	ARL				SDRL			
0.00	250.0	250.0	250.0	250.0	281.6	276.5	273.3	269.4
0.50	25.9	13.3	8.4	5.3	32.5	16.6	10.3	6.3
0.75	8.0	3.9	2.6	1.8	9.9	4.5	2.6	1.6
1.00	3.6	1.9	1.4	1.2	3.7	1.6	0.9	0.5
1.50	1.5	1.1	1.0	1.0	0.9	0.4	0.2	0.1
2.00	1.1	1.0	1.0	1.0	0.4	0.1	0.0	0.0
3.00	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0
4.00	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0

TABLE A.III: ARLs OF THE EWMA AND SYNTHETIC CHARTS, WHERE $n = 3, 5, 7, \text{ AND } 10, \delta_{\text{opt}} = 1, \text{ AND } \text{ARL}_0 = 250$

n	3	5	7	10	3	5	7	10
δ	EWMA				Synthetic			
0.00	252.0	250.5	250.5	250.2	250.0	250.0	250.0	250.0
0.25	52.6	42.7	40.1	36.3	108.1	74.8	55.8	39.9
0.50	13.5	9.7	8.3	6.7	25.9	13.3	8.4	5.3
0.75	6.2	4.3	3.4	2.6	8.0	3.9	2.6	1.8
1.00	3.9	2.6	2.0	1.5	3.6	1.9	1.4	1.2
1.50	2.3	1.5	1.2	1.0	1.5	1.1	1.0	1.0
2.00	1.7	1.1	1.0	1.0	1.1	1.0	1.0	1.0
3.00	1.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0
4.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

TABLE A.IV: SDRLs OF THE EWMA AND SYNTHETIC CHARTS, WHERE $n = 3, 5, 7, \text{ AND } 10, \delta_{\text{opt}} = 1, \text{ AND } \text{ARL}_0 = 250$

n	3	5	7	10	3	5	7	10
δ	EWMA				Synthetic			
0.00	249.6	248.9	249.5	249.4	281.6	276.5	273.3	269.4
0.25	49.5	40.5	38.6	35.2	126.8	87.4	65.0	46.1
0.50	10.6	7.7	6.8	5.7	32.5	16.6	10.3	6.3
0.75	3.8	2.6	2.2	1.7	9.9	4.5	2.6	1.6
1.00	1.9	1.3	1.0	0.8	3.7	1.6	0.9	0.5
1.50	0.8	0.6	0.4	0.2	0.9	0.4	0.2	0.1
2.00	0.6	0.3	0.1	0.0	0.4	0.1	0.0	0.0
3.00	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

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