Horizontal Vibration of a Rigid Circular Foundation on a Transversely Isotropic Half-Space

Hamed Moghaddasi, Mohammad Rahimian, and Ali Khojasteh

***Abstract*—Horizontal vibration of a rigid circular foundation rested on the top of a transversely isotropic half-space is presented. By using fundamental Green's functions and applying Hankel integral transform in the radial direction and Fourier series the problem may be changed to a system of four separate integral equations, which, in turn, are reduced to a pair of Fredholm equation of the second kind. Under dynamic excitation, the related compliance function are numerically evaluated. The present solutions are analytically and numerically in exact agreement with the existing solutions for a half-space with isotropic material.**

***Index Terms*— Horizontal vibration, rigid foundation, dual integral equation, transversely isotropic half -space, green's function.**

I. Introduction

In soil-structure interaction problems, the dynamic compliance of a foundation on a half-space plays a key role in determining the response of the surface structures to dynamic loadings, in particular, seismic excitation and machine vibration. On the dynamic interaction of a rigid disc with an elastic medium, treatments of the surface disc are available as in Bycroft [1], Gladwell [2].The results on the axial, torsional, horizontal, and rocking response of a surface disc can be found in Luco & Westmann [3]. Later, Pak and Saphores [4] presented an analytical formulation for the general problem of a rigid disc embedded in an isotropic half-space under lateral load, which was solely considered earlier. Recently, applications of anisotropic media have been increased in foundation engineering. Based on the differences between isotropic and anisotropic media, wave propagation problem in these two media is significantly different. In addition, deposited soils show an anisotropic behavior such that in application, they need to be modeled as transversely isotropic or orthotropic materials. However the analytical treatment of the problem of vibrations of a circular disc associated with a general anisotropic medium is left unsolved, mainly because of the more complicated nature of its constitutive behavior. In this class, Selvadurai [5] studied rigid disc inclusion in a transversely isotropic full-space for different boundary conditions. Recently, Rahimian et al. [6] studied the Reissner–Sagoci problem for a transversely isotropic half-space. The main concern of the present work is the horizontal vibration of a rigid circular disc on a transversely isotropic half space. It is particularly important in the seismic design of structures, because generally it is the horizontal component of earthquake excitation that governs the seismic response. The coupled partial differential equations are uncoupled with the use of potential functions introduced by Eskandari-Ghadi et al. [7]. Khojasteh et al. [8] obtained fundamental Green's functions for a transversely isotropic elastic half-space subjected to an arbitrary, time-harmonic, finite, buried source. Numerical evaluations for various transversely isotropic materials along with computed functions are graphically illustrated in order to show the effect of different material anisotropy. This rigorous study can lead to more reliable benchmarks for broadly used numerical studies which may be used as a rational basis for developing approximate and more advanced treatments.

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II. Procedure for Mathematical Formulations

Consider a rigid massless disc of radius a on a homogeneous transversely isotropic, elastic half-space. A prescribed time-harmonic horizontal movement, $\Delta\, e^{i\omega t}$, with Δ and ω, being the amplitude and circular frequency of the motion, respectively, is considered for the disc. A relaxed treatment of this mixed boundary-value problem can be stated in terms of the components of the displacement vector u and the Cauchy stress tensor σ as follows:

$$u_r(r,\theta,0) = \Delta\cos(\theta)e^{i\omega t},$$
$$u_\theta(r,\theta,0) = -\Delta\sin(\theta)e^{i\omega t}, \qquad (1)$$
$$r < a$$

$$\sigma_{zi}(r,\theta,0) = 0, \quad i = r,\theta \qquad r > a \qquad (2)$$

$$\sigma_{zr}(r,\theta,0) = -P(r,\theta), \quad \sigma_{z\theta}(r,\theta,0) = -Q(r,\theta) \qquad r < a \qquad (3)$$

Here, $P(r,\theta)$ and $Q(r,\theta)$ denote the components of the unknown tangential contact-load distribution acting on the disc in the radial and angular directions, respectively. It was noticed that all the above equations hold for $0 < \theta < 2\pi$ .For a half-space, the foregoing requirements must be appended by the regularity condition at infinity that $\sigma \to 0$ as $\sqrt{r^2 + z^2} \to \infty$. The equations of motion for a homogeneous transversely isotropic elastic solid in terms of displacements and in the absence of body forces can be found in [8] .In order to uncouple this Equations , a set of complete potential functions F and χ introduced by Eskandari-Ghadi [7] is used. With the aid of F, χ and conditions (3) provide equations required for the solution of the u_r and u_θ in terms of the transform of the Fourier components P_m and Q_m of the contact-load distribution (see [8]).In particular, one may verify that the radial and angular displacements and stress can in general be expressed as

$$u_{r_m} \mp i\, u_{\theta_m} = \int_0^\infty \xi\{\pm\gamma_1(\xi,z)\frac{X_m - Y_m}{2c_{44}} + \gamma_2(\xi,z)\frac{X_m + Y_m}{2c_{44}}\}J_{m\mp1}(r\xi)d\xi \quad (4)$$

$$\sigma_{zr_m} \mp i\,\sigma_{z\theta_m} = \int_0^\infty \xi\{\pm\Pi_1(\xi,z)\frac{X_m - Y_m}{2c_{44}} + \Pi_2(\xi,z)\frac{X_m + Y_m}{2c_{44}}\}J_{m\mp1}(r\xi)d\xi \quad (5)$$

Here,

$$X_m = \tilde{P}_m^{m-1}(\xi) + i\tilde{Q}_m^{m-1}(\xi), Y_m = \tilde{P}_m^{m-1}(\xi) - i\tilde{Q}_m^{m-1}(\xi)$$

In addition $\gamma_1(\xi,z), \gamma_2(\xi,z), \Pi_1(\xi,z)$ and $\Pi_2(\xi,z)$ can be found in [8]. On the account of (1) and the orthogonality of $\{e^{im\theta}\}$, it can be shown that:

$$X_1 = Y_{-1},\ X_{-1} = Y_1,\ X_m = Y_m = 0 \quad m \neq \pm1 \quad (6)$$

By recourse to (4, 5), the remaining four conditions of the mixed boundary-value problem can thus be reduced to

$$\int_0^\infty \frac{1}{\xi}\{\Gamma_1(\xi)X(\xi) + \Gamma_2(\xi)Y(\xi)\}J_0(r\xi)d\xi = \Delta \quad r < a \quad (7)$$

$$\int_0^\infty \frac{1}{\xi}\{\Gamma_2(\xi)X(\xi) + \Gamma_1(\xi)Y(\xi)\}J_2(r\xi)d\xi = 0 \quad r < a \quad (8)$$

$$\int_0^\infty X(\xi)J_0(r\xi)d\xi = 0 \quad r > a \quad (9)$$

$$\int_0^\infty Y(\xi)J_2(r\xi)d\xi = 0 \quad r > a \quad (10)$$

in which:

$$\Gamma_1(\xi) = (\gamma_1(\xi,0) + \gamma_2(\xi,0))\xi,$$
$$\Gamma_2(\xi) = (\gamma_2(\xi,0) - \gamma_1(\xi,0))\xi,$$
$$X(\xi) = \frac{X_1\xi}{2c_{44}}, \quad Y(\xi) = \frac{Y_1\xi}{2c_{44}}$$

$\Gamma_1(\xi)$ and $\Gamma_2(\xi)$ have the properties that:

$$l_1 = \lim_{\xi\to\infty}\Gamma_1(\xi) = \frac{c_{33}((s_1-s_2)^2(s_1+s_2)+s_0(s_1^2+s_2^2)) - 2s_0(c_{13}+2c_{44})}{c_{33}s_0(s_1+s_2)(s_1-s_2)^2} \quad (11)$$

$$l_2 = \lim_{\xi\to\infty}\Gamma_2(\xi) = \frac{c_{33}\left((s_1-s_2)^2(s_1+s_2) - s_0(s_1^2+s_2^2)\right) + 2s_0(c_{13}+2c_{44})}{c_{33}s_0(s_1+s_2)(s_1-s_2)^2} \quad (12)$$

In the above equations, s_1 and s_2 are the roots of the following equation, which in view of the positive definiteness of the strain energy, are not zero or pure imaginary numbers

$$c_{33}c_{44}s^4 + (c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33})s^2 + c_{11}c_{44} = 0 \quad (13)$$

In addition, cij is the elasticity constants of the solid.

III. Reduction of System of Dual Integral Equations

With the aid of Sonine's integrals [9], the system of coupled dual integral equations in (7-10) can alternatively be expressed as:

$$\int_0^\infty \frac{1}{\sqrt{\xi}}\{(1 + H_1(\xi))X(\xi) + \frac{l_2}{l_1}(1 - H_2(\xi))Y(\xi)\}J_{-\frac{1}{2}}(r\xi)d\xi = \delta\sqrt{\frac{2}{\pi r}} \quad r < a \quad (14)$$

$$\int_0^\infty \frac{1}{\sqrt{\xi}}\{(1 - H_2(\xi))X(\xi) + \frac{l_1}{l_2}(1 + H_1(\xi))Y(\xi)\}J_{\frac{3}{2}}(r\xi)d\xi = 0 \quad r < a \quad (15)$$

$$\int_0^\infty \frac{1}{\sqrt{\xi}}X(\xi)J_{-\frac{1}{2}}(r\xi)d\xi = 0 \quad r > a \quad (16)$$

$$\int_0^\infty \frac{1}{\sqrt{\xi}}Y(\xi)J_{\frac{3}{2}}(r\xi)d\xi = 0 \quad r > a \quad (17)$$

where:

$$H_1(\xi) = \frac{\Gamma_1(\xi)}{l_1} - 1, \quad H_2(\xi) = 1 - \frac{\Gamma_2(\xi)}{l_2}, \quad \delta = \frac{\Delta}{l_1}$$

for further reduction, it is useful to define a function θ_A and θ_B through

$$\theta_A(r) = \sqrt{\frac{\pi r}{2}}\int_0^\infty \frac{1}{\sqrt{\xi}}X(\xi)J_{-\frac{1}{2}}(r\xi)d\xi,\ \theta_B(r) = \sqrt{\frac{\pi r}{2}}\int_0^\infty \frac{1}{\sqrt{\xi}}Y(\xi)J_{3/2}(r\xi)d\xi \quad r < a \quad (18)$$

$$\theta_A(r) = \theta_B(r) = 0 \quad r > a$$

By virtue of (18) and some recurrence relations between Bessel functions of different orders, the governing system of coupled dual integral equations can be reduced to a pair of Fredholm integral equations of the second kind

$$\theta_A(r) + \frac{l_2}{l_1}\left[\int_r^a \frac{\theta_B(\rho)}{\rho} d\rho - \theta_B(r)\right] + \int_0^a K_{AA}(r,\rho)\,\theta_A(\rho)d\rho + \int_0^a K_{AB}(r,\rho)\,\theta_B(\rho)d\rho = \delta \tag{19}$$

$$\theta_B(r) + \frac{l_2}{l_1}\left[\frac{1}{r}\int_0^r \theta_A(\rho)\ d\rho - \theta_A(r)\right] + \int_0^a K_{BB}(r,\rho)\,\theta_B(\rho)d\rho + \int_0^a K_{BA}(r,\rho)\,\theta_A(\rho)d\rho = 0 \tag{20}$$

$$K_{AA}(r,\rho) = \sqrt{r\rho}\int_0^\infty\ \xi\ H_1(\xi) J_{-\frac{1}{2}}(r\xi) J_{-\frac{1}{2}}(\rho\xi) d\xi \tag{21}$$

$$K_{BB}(r,\rho) = \sqrt{r\rho}\int_0^\infty\ \xi\ H_1(\xi) J_{\frac{3}{2}}(r\xi) J_{\frac{3}{2}}(\rho\xi) d\xi \tag{22}$$

$$K_{AB}(r,\rho) = K_{BA}(\rho,r) = -\frac{l_2}{l_1}\sqrt{r\rho}\int_0^\infty\ \xi\ H_2(\xi) J_{-\frac{1}{2}}(r\xi) J_{\frac{3}{2}}(\rho\xi) d\xi \tag{23}$$

By utilizing similar relations like Pak and saphores [4] the contact-load distribution in angular and radial directions can be evaluated directly in terms of the solution of the Fredholm equation. By Projecting the radial and angular contact-load distributions *P* and *Q* onto the Cartesian frame, one finds that the rectangular components of the force F required to achieve the disc displacement Δ are given by

$$F_x = 8c_{44}\int_0^a \theta_A(\rho)\,d\rho \tag{24}$$

F_y is identically zero as can be expected from the symmetry of the problem. Also, It can be rewritten in terms of the horizontal impedance which is defined as:

$$K_{HH} = \frac{F_x}{c_{44}a\Delta} \tag{25}$$

It may also express into the dynamic horizontal compliance, which is the ratio of Δ to F_x.

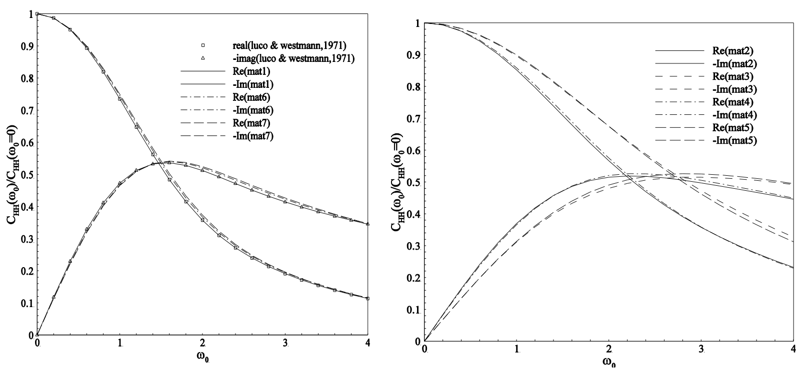
IV. Numerical Results and Discussion

In the previous sections, the Fredholm integral equations were expressed in terms of θ. it is not easy to deal with the dual integral equations analytically. For this reason, numerical solutions of the integral equation can be obtained by standard quadrature methods. On tackling with (21-23) , some special considerations are needed due to the presence of singularities within the range of integration including branch points. In addition, some functions in $\gamma_1(\xi,z), \gamma_2(\xi,z)$ yields pole at ξ_R which corresponds to Rayleigh wave number. The numerical results presented here are dimensionless by using a non-dimensional frequency as: $\omega_0 = a\omega\sqrt{\rho_s/c_{44}}$. In the forgoing equations, ρ_s stands for soil density. To understand the effect of anisotropy of the materials on the interaction between the two media, several synthetic types of isotopic (mat 1) and transversely isotropic materials (mat 3–7) are used. The material properties are given in Table 1, where E and E′ are the Young's modules in the plane of isotropy and perpendicular to it; ν is Poison's ratio that characterize the effect of horizontal strain on the complementary vertical strain; ν′ is the Poisson's ratio which characterize the effect of vertical strain on the horizontal one; and G' is the shear modulus for the plane normal to the plane of isotropy. In defining these materials, the positive-definiteness of strain energy that observe the following constraints for material constants c_{ij}, have been checked [10]

$$c_{11} > |c_{12}|\,, \qquad (c_{11}+c_{12})c_{33} > 2c_{13}{}^2\,, \qquad c_{44} > 0 \tag{26}$$

TABLE I: Properties of Synthetic Materials.

mat	*E*	*E'*	*G*	*G'*	*ν, ν'*	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	c_{66}
1	5	5	2	2	0.25	6	2	2	6	2	2
2	10	5	4	2	0.25	14	6	5	7.5	2	4
3	15	5	6	2	0.25	26	14	10	10	2	6
4	5	5	2	1	0.25	6	2	2	6	1	2
5	5	5	2	0.67	0.25	6	2	2	6	0.67	2
6	5	10	2	2	0.25	5.6	1.6	1.8	10.9	2	2
7	5	15	2	2	0.25	5.5	1.5	1.8	15.9	2	2



a) b)

Fig. 1. Compliance function in terms of dimensionless frequency for different isotropic and transversely isotropic materials .a)mat1,6-7 b) mat2-5.

The impedance/compliance function is a very significant parameter in the subject of soil–structure-nteraction. Figure 1a shows the real and imaginary parts of the horizontal compliance function obtained from the present study and the respective results from Luco & westmann [3] for a high range of dimensionless frequency. There exists an excellent agreement between these two results, as seen in this figure, which demonstrates the accuracy of the numerical evaluation in different steps.Also, this figure contains tow transversely isotropic materials with different E' value, it was deduced from this figure that by increasing E' value, results similar to material 1 (isotropic material) were obtained e.g. E' value has only minor effects on form of compliance function.Figures 1b provide the horizontal compliance function in terms of dimensionless frequency for rest of transversely isotropic materials(mat 2-5). The horizontal compliance function for the rigid plate is affected by changing the E and G' value.It was worth mentioning that Dynamic compliances computed in the present study are in the dimensionless form $C_{HH}(\omega_0)/C_{HH}(\omega_0=0)$ where $C_{HH}(\omega_0=0)$ is the horizontal compliance of a rigid circular disc on an transversely isotopic half-space under static loading.

V. Conclusion

One of the key steps in the dynamic analysis of foundation soil system under seismic or machine type loading is to determine the dynamic impedance/compliance functions associated with rigid foundations. in the recent decade, There was demands to model more accurately the mechanical behavior of natural geological deposits as well as many composites and engineered materials which often are transversely isotropic. Hence, in this paper a mathematical formulation for the horizontal vibration of a rigid circular disc on a transversely isotropic half space has been presented. To facilitate its direct engineering applications as well as its use as a Green's function in other boundary-value problems, the dynamic horizontal compliances for different transversely isotropic materials are included. Also, it was evident that between material coefficients in transversely isotropic media, the shear modulus for the plane normal to the plane of isotropy and the Young’s modules in the plane of isotropy would have the most significant effect on lateral load transfer process.

References

[1] G. N. Bycroft, “Forced vibrations of a rigid circular footing on a semi-infinite elastic space and on a elastic stratum,” *Phil. Trans. R. Soc. Lond.* 1956, vol. 248A, no. 948, pp. 3327-3368.

[2] G. M. L. Gladwell, “Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid,” *Int. J. Engng. Sci.* 1968, vol. 6 , no. 10, pp. 591–607.

[3] J. E. Luco and R. A. Westmann, “Dynamic response of circular footing,” *J. Engng. Mech. Div.* 1971, vol. 97, no. 5, pp. 1381-1395.

[4] R. Y. S. Pak and J. Saphores, “Lateral translation of a rigid disc in a semi-infinite solid,” *Q. J. Mech. Appl. Math*. 1992, vol. 42, pp. 435-449.

[5] A. P. S. Selvadurai, “Asymmetric displacements of a rigid disc inclusion embedded in a transversely isotropic elastic medium of infinite extent,” *Int. J. Engng. Sci.* 1980, vol. 18, no. 7, pp. 979–986.

[6] M. Rahimian, A. K. Ghorbani-Tanha, and M. Eskandari-Ghadi, “The Reissner–Sagoci problem for a transversely isotropic half-space,” *Int. J. Numer. Anal. Meth. Geomech*. 2006, vol. 30, no. 11, pp. 1063–1074.

[7] M. Eskandari-Ghadi, “A complete solutions of the wave equations for transversely isotropic media,” *Journal of Elasticity*. 2005, vol. 81, no. 1, pp. 1–19.

[8] A. Khojasteh, M. Rahimian, M. Eskandari, and R. Y. S. Pak, “Asymmetric wave propagation in a transversely isotropic half-space in displacement potentials,” *Int. J. Engng. Sci.* 2008, vol. 46, no. 7, pp. 690–710.

[9] B. Noble, “The solution of bessel function dual integral equations by a multiplying-factor method,” *Proc. Cambridge. Phil. Soc.* 1963, vol. 59, no. 2, pp. 351–371.

[10] R. G. Payton. *Elastic wave propagation in transversely isotropic media*. the Netherlands: Martinus, Nijhoff. 1983