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A Computer-Aided Approach for Routh-Padé Approximation of SISO Systems Based on Multi-objective Optimization

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***Abstract*—A multi-objective optimization based computer-aided method to derive a reduced order (rth-order) approximant for given (stable) SISO linear continuous-time system is presented. In this method, stability and the first r time moments/Markov parameters are preserved as well as the errors between a set of subsequent time moments/Markov parameters of the system and those of the model are minimized. The method is useful as it alleviates the problems of deciding the values of number of error functions to be minimized and values of weights on the errors in arriving at good approximants.**

***Index Terms*—Model reduction,Padé approximation, Routh criterion.**

I. INTRODUCTION

The usefulness of techniques for deriving low-order approximations of high-order systems has already been accepted due to the advantages of reduced computational effort and increased understanding of the original system. Consequently, a large number of time-domain and frequency-domain system simplification techniques have been developed to suit different requirement. Amongst them, a frequency domain method is Padé approximation in which 2*r* terms of the power series expansion (time moments) of the high-order (*n*th-order) transfer function $G_n(s)$ are fully retained in low-order (*r*th-order) model $G_r(s)$. The Padé approximation does not guarantee the stability of the reduced-order model. To overcome the problem of stability, several stable reduction methods such as Routh approximation [1-3], the Hurwitz polynomial approximation [4], the stability equation method [5] and the method using Michailov stability criterion [6] have been proposed. The Routh approximation [1-3] has the drawback of matching only the first *r* time moments ($t_1, t_2, ..., t_r$) of $G_n(s)$ to the respective time moments ($\hat{t}_1, \hat{t}_2, ..., \hat{t}_r$) of $G_r(s)$ (in recent years the extension of Routh approximation techniques [1-3] to interval systems has attracted the attention of many researchers [7-11].

Later Shamash [12] considered the effect of including some Markov parameters ($M_1, M_2, ...$) along with time moments, which is generally essential to ensure both initial and steady state response approximation. However, the technique of [12] is again confined to matching of only *r* terms (α time moments and β Markov parameters, where $\alpha + \beta = r$). Several variants of Routh approximation were subsequently reported [13-16]; however, they again remain confined to only *r* terms matching for the purpose of preserving stability, a task which can be achieved arbitrarily [17,18]. Note that infinite numbers of stable models can be constructed if the objective is to match only *r* terms [18].

Thus, the basic problem is to match or near match a few terms in excess of *r* terms while preserving stability [19,20]. Some attempt was made previously [21-23] to partially solve this problem. Singh [22] suggested a technique based on the successive variances of the model. The method [22] requires the determination of the stability region in terms of the free parameters. A modification of above technique was given by Lepeschy and Viaro [23]. Other closely related problems have also received attention [24-36]. Recently, geometric programming based (computer-oriented) methods [37,38] for the solution of the Routh-Padé approximation problem are presented. In these methods [37,38], first *r* time moments/Markov parameters are fully retained and the sum of the weighted squares of errors between a set of subsequent time moments/Markov parameters of the system and those of the model are minimized while preserving stability. These methods [37,38] have the drawback that the question of finding some means (free of hit and trial) of deciding the values of the number of time moments/Markov parameters (say *m*) to be matched or near-matched and the weights to correspond to assured substantial improvement in system approximation as well as the question of establishing the existence of such values are left unresolved.

In this note, a nonlinear programming based (computer-oriented) method for the solution of Routh-Padé approximation problem is presented. The method is essentially a multi-objective optimization procedure in which not only stability is preserved and the first *r* terms of the power series expansion of $G_n(s)$ are fully retained but also the errors between a set of subsequent time moments/Markov parameters of the system and those of the model are minimized. In this method, the objective

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functions are ranked in order of importance. The optimum solution is then found by minimising the objective functions starting with the most important one and proceeding according to the order of importance of the objectives. This alleviates the problem of finding *m* and weights. For the solution of this problem, Luss and Jaakola [39] algorithm is proposed.

II. Model Reduction Problem Viewed as a Multi-Objective Optimization Problem

Consider a single-input-single-output system described by the transfer function

$$G_n(s) = \frac{a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}{s^n + b_1 s^{n-1} + \ldots + b_n} \quad (1)$$

$$= t_1 + t_2 s + \ldots + t_n s^{n-1} + \ldots \quad (2)$$

(expansion around $s = 0$)

$$= M_1 s^{-1} + M_2 s^{-2} + \ldots + M_n s^{-n} + \ldots \quad (3)$$

(expansion around $s = \infty$)

The problem is to determine its stable reduced-order (*r*th-order) approximant

$$G_r(s) = \frac{\hat{a}_1 s^{r-1} + \hat{a}_2 s^{r-2} + \ldots + \hat{a}_r}{s^r + \hat{b}_1 s^{r-1} + \ldots + \hat{b}_r} \quad (4)$$

$$= \hat{t}_1 + \hat{t}_2 s + \ldots + \hat{t}_r s^{r-1} + \ldots \quad (5)$$

$$= \hat{M}_1 s^{-1} + \hat{M}_2 s^{-2} + \ldots + \hat{M}_r s^{-r} + \ldots \quad (6)$$

A. Formulation of the objective function

The formulation of the multiobjective optimization problem will be explained for *r* being even. Formulation for r being odd can be done in a similar way. It is easy to verify that for r even, the following equations hold true:

$$\left.\begin{aligned} \hat{a}_{r+1-i} &= \sum_{j=1}^{i} \hat{t}_j \hat{b}_{r-i+j} \\ \hat{a}_i &= \sum_{j=1}^{i} \hat{M}_j \hat{b}_{i-j} \end{aligned}\right\} \quad i = 1, \ldots, \frac{r}{2} \quad (7)$$

$$\left.\begin{aligned} \hat{t}_i &= \left(-\sum_{j=1}^{i-1} \hat{t}_j \hat{b}_{r-i+j} + \sum_{j=1}^{r+1-i} \hat{M}_j \hat{b}_{r+1-i-j}\right)\hat{b}_r^{-1} \\ \hat{M}_i &= \sum_{j=1}^{r+1-i} \hat{t}_j \hat{b}_{i-1+j} - \sum_{j=1}^{i-1} \hat{M}_j \hat{b}_{i-j} \end{aligned}\right\} \quad i = \frac{r}{2}+1, \frac{r}{2}+2, \ldots \quad (8)$$

($\hat{b}_0 = 1; \hat{b}_i = 0$ for $i \notin \{0, \ldots, r\}$; $\hat{t}_i, \hat{M}_i = 0$ for $i < 1$).

We seek a stable model for which *r* equations given by

$$\left.\begin{aligned} \hat{t}_i - t_i &= 0 \\ \hat{M}_i - M_i &= 0 \end{aligned}\right\} i = 1, \ldots, \frac{r}{2} \quad (9)$$

are satisfied ,which implies, from (7),

$$\left.\begin{aligned} \hat{a}_{r+1-i} &= \sum_{j=1}^{i} t_j \hat{b}_{r-i+j} \\ \hat{a}_i &= \sum_{j=1}^{i} M_j \hat{b}_{i-j} \end{aligned}\right\} \quad i = 1, \ldots, \frac{r}{2}. \quad (10)$$

There exist an infinite number of stable models for which (10) is satisfied [18]. This arbitrariness in stability preservation can be exploited to minimize square of the errors of matching of some additional time moments and Markov parameters of the system with those of the model, namely, to minimize objective functions $Z_1, Z_2, \ldots, Z_k$ given by

$$\left.\begin{aligned} Z_1 &= \left(1 - \frac{\hat{t}_{\frac{r}{2}+1}}{t_{\frac{r}{2}+1}}\right)^2 \\ Z_2 &= \left(1 - \frac{\hat{M}_{\frac{r}{2}+1}}{M_{\frac{r}{2}+1}}\right)^2 \\ &\vdots \\ Z_{k-1} &= \left(1 - \frac{\hat{t}_{\frac{r}{2}+i}}{t_{\frac{r}{2}+i}}\right)^2 \\ Z_k &= \left(1 - \frac{\hat{M}_{\frac{r}{2}+i}}{M_{\frac{r}{2}+i}}\right)^2 \end{aligned}\right\} \quad i = 1, \ldots, m \quad (11)$$

where *k* denotes the number of objective functions to be minimised. Using (8) subject to (9), (11) can be expressed as

$$\left.\begin{aligned} Z_1 &= f(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r) \\ Z_2 &= f(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r) \\ &\vdots \\ Z_k &= f(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r) \end{aligned}\right\} \quad (12)$$

B. Formulation of the stability constraints

Now following [40], the denominator polynomial of (4) can be expressed as

$$s^r + \hat{d}_1 s^{r-1} + (\hat{d}_2 + \hat{d}_3 + ... + \hat{d}_r) s^{r-2} + \hat{d}_1(\hat{d}_3 + \hat{d}_4 + ... + \hat{d}_r) s^{r-3}$$
$$+ [\hat{d}_2(\hat{d}_4 + \hat{d}_5 + .. + \hat{d}_r) + \hat{d}_3(\hat{d}_5 + \hat{d}_6 + .. + \hat{d}_r) + \hat{d}_4(\hat{d}_6 + \hat{d}_7 + .. + \hat{d}_r) + ..$$
$$+ \hat{d}_{r-2}\hat{d}_r]\, s^{r-4} + ... + \hat{d}_{1+q}\hat{d}_{3+q}...\hat{d}_{r-2}\hat{d}_r \quad (13)$$

which is constructed by taking the coefficients of the first two rows of the Routh array with the elements of its first column given by

$$1, \hat{d}_1, \hat{d}_2, \hat{d}_1\hat{d}_3, \hat{d}_2\hat{d}_4, \hat{d}_1\hat{d}_3\hat{d}_5, ..., \hat{d}_{1+q}\hat{d}_{3+q}...\hat{d}_{r-2}\hat{d}_r \quad (14)$$

where q =1 for r even and q =0 for r odd. By setting

$$\hat{d}_1\hat{b}_1^{-1} = 1, (\hat{d}_2 + \hat{d}_3 + ... + \hat{d}_r)\hat{b}_2^{-1} = 1,$$
$$\hat{d}_1(\hat{d}_3 + \hat{d}_4 + ... + \hat{d}_r)\hat{b}_3^{-1} = 1, ..., (\hat{d}_{1+q}\hat{d}_{3+q}..\hat{d}_{r-2}\hat{d}_r)\hat{b}_r^{-1} = 1 \quad (15)$$

(15) is matched with the denominator polynomial of the model in (4), namely, with

$$\hat{b}_r + \hat{b}_{r-1}s + ... + \hat{b}_1 s^{r-1} + s^r \quad (16)$$

and the necessary and the sufficient condition that all the roots of (16) be strictly in the left half plane is [40]

$$\hat{d}_1 > 0, \hat{d}_2 > 0, ..., \hat{d}_r > 0 \quad (17a)$$

which, of course, implies

$$\hat{b}_1 > 0, \hat{b}_2 > 0, ..., \hat{b}_r > 0 \quad (17b)$$

Note that, for a given r, $\hat{b}_i$, $i = 1,...,r$, can easily be expressed in terms of $\hat{d}_i$, $i = 1,...,r$, by constructing an inverse Routh array (i.e., with the element of its first column given by (14)) in a manner analogous to [40].Thus, pertaining to r=4, (15) becomes

$$\hat{b}_1 = \hat{d}_1,\ \hat{b}_2 = \hat{d}_3 + \hat{d}_3 + \hat{d}_4,\ \hat{b}_3 = \hat{d}_1(\hat{d}_3 + \hat{d}_4),$$
$$\hat{b}_4 = \hat{d}_2\hat{d}_4. \quad (18)$$

C. *Problem formulation*

The problem is to minimize (12) subject to (15) and (17). This problem is converted to a multi-objective optimization problem and is stated as follows:

$$\text{Find } \mathbf{B} = \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_r \end{Bmatrix}$$
$$\text{which minimizes } Z_1, Z_2, ..., Z_k \text{ subject to (15) and (17).} \quad (19)$$

For minimization of error-functions, objective functions are arranged in order of importance. Let the subscript of the objective functions indicate not only the objective function number but also the priorities of the objective functions. Thus, the minimization Z_1 is most important and the minimization of Z_k is least important. The first problem is formulated as

$$\left.\begin{array}{l} \text{minimize } Z_1 \\ \text{subject to (15) and (17).} \end{array}\right\} \quad (20)$$

and its solution $\mathbf{B}_1^*$ and Z_1^* is obtained. Then the second problem is formulated as

$$\left.\begin{array}{l} \text{minimize } Z_2 \\ \text{subject to (15) and (17)} \\ \text{and} \\ Z_1 = Z_1^* \end{array}\right\} \quad (21)$$

The solution of this problem is obtained as $\mathbf{B}_2^*$ and Z_2^*. This procedure is repeated until all the k objective functions have been considered. The jth problem (j=1,…,k) is given by

$$\left.\begin{array}{l} \text{minimize } Z_j \\ \text{subject to (15) and (17)} \\ \text{and} \\ Z_l = Z_l^* \quad l = 1,2,..., j-1 \end{array}\right\} \quad (22)$$

and its solution is found as $\mathbf{B}_j^*$ is found as Z_j^*. Finally solution obtained at the end (i.e., $\mathbf{B}_k^*$) is taken as the desired solution $\mathbf{B}^*$ of the original multi-objective optimization problem.

III. Application of An Algorithm Due to Luss And Jaakola

Owing to the strictly positive nature of $\hat{d}_i$'s and $\hat{b}_i$'s and the particular form of the objective function in (12), the Luss and Jaakola algorithm is found to be particularly useful for solving the above-stated problem. In this algorithm [39], an initial point and an initial interval are chosen at random. Depending on the function values at a number of random points in the interval, the search interval is reduced at every iteration by a constant factor. It has successfully produced improved approximants relative to the conventional Routh approximants [1-3]. Although, the multi-objective optimization problem is formulated for a pre-chosen value of m, our numerical experience shows that substantial improvement is obtained with lesser number of iterations. The following criterion can used to terminate the iterative process:

$$\left|\mathbf{B}_{\mathbf{k+1}}^* - \mathbf{B}_{\mathbf{k}}^*\right| \le \varepsilon \quad (23)$$

IV. Examples

The step-by-step procedure to obtain reduced-order model is explained with the help of the examples presented below.

***Example* 1**

Consider a stable third-order system [19]

$$G(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2} \quad (24)$$
$$(t_1 = 1, t_2 = 0.5, t_3 = 0.75;$$
$$M_1 = 8, M_2 = -26, M_3 = 66).$$

Suppose a second-order approximant (r=2) is required. The approximant can systematically be arrived at by following the steps given below.

***Step* 1.** From the requirement of the first r terms matching (see (10)), one has

$$\left.\begin{aligned}\hat{a}_2 &= t_1\hat{b}_2\\ \hat{a}_1 &= M_1\end{aligned}\right\}\Rightarrow \quad \left\{\begin{aligned}\hat{a}_2 &= \hat{b}_2\\ \hat{a}_1 &= 8.\end{aligned}\right. \tag{25}$$

***Step* 2.** From (8) together with (9), one obtains

$$\hat{t}_2 = (-\hat{b}_1+8)\hat{b}_2^{-1} \qquad \hat{M}_2 = \hat{b}_2 - 8\hat{b}_1$$

$$\hat{t}_3 = -(1+\hat{t}_2\hat{b}_1)\hat{b}_2^{-1} = -(1+8\hat{b}_1\hat{b}_2^{-1} - \hat{b}_1^2\hat{b}_2^{-1})\hat{b}_2^{-1}$$

$$\hat{M}_3 = -(8\hat{b}_2 + \hat{M}_2\hat{b}_1) = -(\hat{b}_1\hat{b}_2 - 8\hat{b}_1^2 + 8\hat{b}_2). \tag{26}$$

***Step* 3.** Taking m=2 ($\Rightarrow k=4$) and using (23), the objective functions (11), are constructed as

$$\left.\begin{aligned}
Z_1 &= (1-\frac{\hat{t}_2}{t_2})^2 = (1-\frac{8\hat{b}_2^{-1}-\hat{b}_1\hat{b}_2^{-1}}{0.5})^2\\
Z_2 &= (1-\frac{\hat{M}_2}{M_2})^2 = (1-\frac{-\hat{b}_2^{-1}-8\hat{b}_1\hat{b}_2^{-1}+\hat{b}_1^2\hat{b}_2^{-1}}{0.75})^2\\
Z_3 &= (1-\frac{\hat{t}_3}{t_2})^2 = (1-\frac{\hat{b}_2-8\hat{b}_1}{26})^2\\
Z_4 &= (1-\frac{\hat{M}_2}{M_3})^2 = (1-\frac{-\hat{b}_1\hat{b}_2+8\hat{b}_1^2-\hat{b}_2}{66})^2
\end{aligned}\right\} \tag{27}$$

***Step* 4.** The constraints (see (15) and (17)) are

$$\left.\begin{aligned}\hat{d}_1\hat{b}_1^{-1} &= 1\\ \hat{d}_2\hat{b}_2^{-1} &= 1\end{aligned}\right\} \tag{28}$$

$$\hat{d}_1 > 0, \hat{d}_2 > 0, \hat{b}_1 > 0, \hat{b}_2 > 0. \tag{29}$$

***Step* 5.** The first problem (see (20)) takes the following form

$$\left.\begin{aligned}&\text{minimize } Z_1 = (1-\frac{8\hat{b}_2^{-1}-\hat{b}_1\hat{b}_2^{-1}}{0.5})^2\\ &\text{subject to (29) and (30)}\end{aligned}\right\} \tag{30}$$

***Step* 6.** Luss-Jaakola algorithm [39] is used to obtain optimal values $\mathbf{B}_1^*$ and Z_1^* satisfying (30). Starting with various initial conditions $\hat{d}_1^0, \hat{d}_2^0, \hat{b}_1^0, \hat{b}_2^0$, the algorithm converges to the following optimal solution:

$$\begin{Bmatrix}5.3118\\5.3763\end{Bmatrix}, Z_1^* = 0.0 \tag{31}$$

The second problem is formulated as

$$\left.\begin{aligned}&\text{minimize}\\ &Z_2 = (1-\frac{-\hat{b}_2^{-1}-8\hat{b}_1\hat{b}_2^{-1}+\hat{b}_1^2\hat{b}_2^{-1}}{0.75})^2\\ &\text{subject to (29) and (30) and}\\ &Z_1 = Z_1^*\end{aligned}\right\} \tag{32}$$

The algorithm converges to the following optimal solution:

$$\mathbf{B}_2^* = \begin{Bmatrix}4.3202\\7.3595\end{Bmatrix}, \quad Z_2^* = 0.0 \tag{33}$$

Now the third problem is formulated as follow

$$\begin{aligned}&\text{minimize } Z_3 = (1-\frac{\hat{b}_2-8\hat{b}_1}{26})^2,\\ &\text{subject to (29) and (30) and } Z_1 = Z_1^*,\ Z_2 = Z_2^*\end{aligned} \tag{34}$$

The following optimal solution is obtained:

$$\mathbf{B}_3^* = \begin{Bmatrix}4.3235\\7.3588\end{Bmatrix}, \quad Z_3^* = 0.0 \tag{35}$$

It is found that there is no significant change in optimal solutions as obtained from second to third problem formulation (see (34), (35)) and the algorithm terminates at second problem formulation. This is also confirmed by integral squared error (ISE) $= \int_0^\infty [y(t)-\hat{y}(t)]^2\,dt$ where $y(t)$ and $\hat{y}(t)$ denote the output responses of $G_n(s)$ and $G_r(s)$. shown in Table 1. Table 1 shows, for a typical initial condition, the progress of the algorithm. Thus the desired values of

$$\mathbf{B}^* \text{ are } \hat{b}_1 = 4.3202, \quad \hat{b}_2 = 7.3595. \tag{36}$$

TABLE I.

Problem No.	Objective function	Constraints	Initial conditions	Optimal B	
1st	$Z_1 = (1-\frac{\hat{t}_2}{t_2})^2$	(28) and (29)	$\hat{b}_1^0 = \hat{d}_1^0 = 5$, $\hat{b}_2^0 = \hat{d}_2^0 = 5$	$\mathbf{B}_1^* = \begin{Bmatrix} 5.3118 \\ 5.3763 \end{Bmatrix}$	$Z_1^* = 0.0$ / ISE = 0.190
2nd	$Z_2 = (1-\frac{\hat{M}_2}{M_2})^2$	(28) and (29) $(1-\frac{\hat{t}_2}{t_2})^2 = Z_1^*$	$\hat{b}_1^0 = \hat{d}_1^0 = 5.3118$, $\hat{b}_2^0 = \hat{d}_2^0 = 5.3763$	$\mathbf{B}_2^* = \begin{Bmatrix} 4.3202 \\ 7.3595 \end{Bmatrix}$	$Z_2^* = 0.0$ / ISE = 0.117
3rd	$Z_3 = (1-\frac{\hat{t}_3}{t_2})^2$	(28) and (29) $(1-\frac{\hat{t}_2}{t_2})^2 = Z_1^*$, $(1-\frac{\hat{M}_2}{M_2})^2 = Z_2^*$	$\hat{b}_1^0 = \hat{d}_1^0 = 4.3202$, $\hat{b}_2^0 = \hat{d}_2^0 = 7.3595$	$\mathbf{B}_3^* = \begin{Bmatrix} 4.3202 \\ 7.3595 \end{Bmatrix}$	$Z_2^* = 0.0$ / ISE = 0.117

***Step* 7.** Using (25), the values of $\hat{a}_1, \hat{a}_2$ are obtained:

$$\hat{a}_1 = 8.0, \quad \hat{a}_2 = 7.3595 \tag{37}$$

Step 8. The model is now identified as

$$G_2(s) = \frac{8.0s + 7.3595}{s^2 + 4.3202s + 7.3595} \tag{38}$$

The step responses (responses to unit step input) of (24) and (38) are plotted in Fig. 1. Fig. 1 also depicts the step response of

$$G_2(s) = \frac{8.0s + 8.129044}{s^2 + 4.30713s + 8.129044} \tag{39}$$

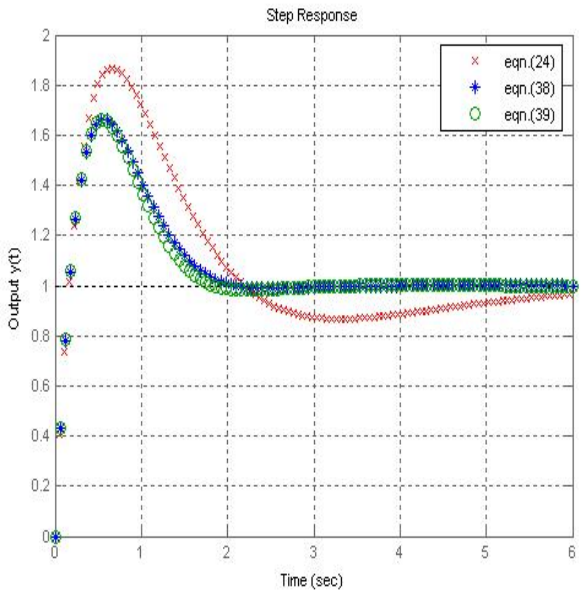

Fig. 1. Step responses of original system (24) and its reduced-order approximants.

The model in (39) is obtained by an approach due to Singh [37]. Fig. 1 reveals that both (38) and (39) are good approximants to (24) but (38) is slightly better than (39) as ISE corresponding to (38) and (39) are found to be 0.11 and o.14. The problems of finding the values of m and weights are also eliminated in arriving at (38). Further, (39) is obtained by matching or near matching of (r+2$\Rightarrow k = 4$) terms (time moments/Markov parameters) while in arriving at (38), only (r+1$\Rightarrow k = 3$) terms are used thus alleviating complexity and convergence problem.

***Example* 2**

Suppose for a fourth-order system given by [23]

$$G(s) = \frac{267s^3 + 527s^2 + 385s + 100}{s^4 + 4s^3 + 6s^2 + 4s + 1} \tag{40}$$

$$(t_1 = 100, t_2 = -15, t_3 = -13, t_4 = 9;$$
$$M_1 = 267, M_2 = -541, M_3 = 947, M_4 = -1510)$$

a second-order approximant is to be obtained.

***Step* 1.** From (10),

$$\left.\begin{matrix} \hat{a}_2 = t_1\hat{b}_2 \\ \hat{a}_1 = M_1 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \hat{a}_2 = 100\hat{b}_2 \\ \hat{a}_1 = 267. \end{matrix}\right. \tag{41}$$

***Step* 2.** From (8) together with (9), taking

m=2 (here $\hat{t}_1 = t_1, \hat{M}_1 = M_1, \hat{t}_2 = t_2$),

$$\hat{t}_2 = (-100\hat{b}_1 + 267)\hat{b}_2^{-1}$$
$$\hat{M}_2 = 100\hat{b}_2 - 267\hat{b}_1 \tag{42}$$
$$\hat{t}_3 = (-100 + t_2\hat{b}_1)\hat{b}_2^{-1}$$

As further steps, using the algorithm [39], the values of $\hat{b}_1, \hat{b}_2$, corresponding to minimum value of (11), with m=2 subject to the constraints (see (15) and (17)), are found to be $\hat{b}_1$=3.100, $\hat{b}_2$=3.042, and, after obtaining $\hat{a}_1$, $\hat{a}_2$, from $\hat{t}_1 = t_1$, $\hat{M}_1 = M_1$, the model takes the form $G_2(s)$ =

$$\frac{267s + 304.42}{s^2 + 3.100s + 3.0442} \tag{43}$$

Table 2 shows the progress of the algorithm starting from typical initial conditions.

TABLE II.

Problem No.	Objective function	Constraints	Initial conditions	Optimal **B**	
1st	$Z_1 = (1 - \frac{\hat{t}_2}{t_2})^2$	(28) and (29)	$\hat{b}_1^0 = \hat{d}_1^0 = 3$ $\hat{b}_2^0 = \hat{d}_2^0 = 3$	$\mathbf{B}_1^* = \begin{Bmatrix} 3.1163 \\ 2.9759 \end{Bmatrix}$	$Z_1^* = 0.0/$ ISE = 2.35
2nd	$Z_2 = (1 - \frac{\hat{M}_2}{M_2})^2$	(28) and (29) $(1 - \frac{\hat{t}_2}{t_2})^2 = Z_1^*$	$\hat{b}_1^0 = \hat{d}_1^0 = 3.1163$ $\hat{b}_2^0 = \hat{d}_2^0 =2.9759$	$\mathbf{B}_2^* = \begin{Bmatrix} 3.100 \\ 3.0442 \end{Bmatrix}$	$Z_2^* = 0.0/$ ISE = 0.145
3rd	$Z_3 = (1 - \frac{\hat{t}_3}{t_3})^2$	(28) and (29) $(1 - \frac{\hat{t}_2}{t_2})^2 = Z_1^*$ $(1 - \frac{\hat{M}_2}{M_2})^2 = Z_2^*$	$\hat{b}_1^0 = \hat{d}_1^0 = 3.100$ $\hat{b}_2^0 = \hat{d}_2^0 =3.0442$	$\mathbf{B}_3^* = \begin{Bmatrix} 3.100 \\ 3.0442 \end{Bmatrix}$	$Z_2^* = 0.0/$ ISE = 0.145

For comparison, the model as obtained by technique in [37] is presented as follows:

$$G_2(s) = \frac{267s + 812.9044}{s^2 + 3.100s + 8.129044} \tag{44}$$

Clearly, the relative superiority of (43) over (44) is for the same reason as explained for Example 1 discussed above, namely, that problem of choosing the values of m and weights are eliminated. This is further confirmed as the ISE values corresponding to(43) are less in comparison to (44).

V. Conclusions

In this paper, the problem of finding Routh-Padé approximants has been viewed as a multi-objective optimization problem. It is shown that, using Luss-Jaakola algorithm [39], the denominator of the model can be chosen so as to minimize errors between the (r+1)th and the subsequent time moments and Markov parameters of the model and the corresponding time moments and Markov parameters of the system while preserving stability. Having obtained the denominator in this manner, the numerator parameters can be determined in the usual manner, namely, by fully retaining the first r time moments/Markov parameters of the system. The present approach, therefore, leads to an improved approximant.

The question of finding some means (free of hit and trial) of deciding the values of m and weights to correspond to assured substantial improvement in system approximation as well as the question of establishing the existence of such values which were left unresolved in [37,38], are eliminated by ranking the objective functions in order of priority. It may be of interest to note that by giving different priorities to the approximation at low frequencies (time moments) and high frequencies (Markov parameters), the user has the flexibility of obtaining a number of stable models of the same order so that he may choose the most suitable model for his purpose.

The possible extension of the present approach using concept of genetic algorithm with Parato curve [41] to arrive at some partial solution to this problem is open to investigation.

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